

Research Paper

## Lyapunov-based Model for Modern Stabilization Systems of All-wheel Drive Vehicles

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### Abstract

The study develops a mathematical model of vehicle stability with torque redistribution, aimed at ensuring guaranteed stabilization under non-stationary conditions. Unlike existing methods, the approach combines Lyapunov functions with bifurcation analysis to derive analytical stability criteria for vehicles with mechanical differentials and enables the synthesis of adaptive control strategies that integrate differential locking, wheel braking, and dynamic torque redistribution with formal stability guarantees. The model provides accurate calculations of torque redistribution to the inner or outer wheels during vehicle oversteer or understeer, respectively, ensuring motion stabilization and preserves stability even under sharp steering inputs, as confirmed by phase portraits and transient response analyses. The proposed model was implemented and verified. The model can be incorporated into active safety systems of wheeled vehicles to enhance stability on complex surfaces, reduce computational requirements, and ensure compatibility with existing mechanical drivetrains.

**Keywords:** Vehicle stability; Torque redistribution; Lyapunov functions; Stabilization system design; All-wheel-drive vehicles

## 1. Introduction

All-wheel-drive (AWD) wheeled vehicles play a critical role in various areas such as transport logistics, agriculture, specialized machinery, amphibious vehicle complexes, etc., where high mobility and stability on deformable or uneven surfaces are essential. During curvilinear motion, especially on terrains with varying adhesion properties, instability may arise due to uneven torque distribution between axles and wheels [1], [2], [3], [4]. Ensuring stable movement in such conditions is a fundamental challenge for the design of modern AWD systems [5], [6], [7], [8]. Addressing this challenge requires constant improvement of the existing approaches and development of new ones to enhance the efficiency of stabilization systems, since conventional solutions – such as electronic stability programs (ESP) or simple differential locking [8], [9], [10], [11], [12] – usually do not

adequately capture the dynamic variations in road/tire interaction and inertia loads.

Researchers are actively investigating torque control methods (torque vectoring, TV) across different vehicle types. A review of the current literature reveals two main research directions. The first group of studies [13], [14], [15], [16], [17], [18], [19] focuses on electric vehicles with independent wheel drives, where torque control can be programmed. For instance, Rill [13] examined torque optimization to reduce cornering resistance by redistributing torque to the outer wheels, although the analysis was limited to stationary modes. Guo *et al.* [15] proposed a nonlinear model predictive control (NMPC) approach based on Lyapunov equations, which guarantees stability but requires substantial computational resources. Diez *et al.* [16] implemented a simplified form of inter-axle torque redistribution. Other works on energy



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efficiency [17], [18] highlight that optimal torque distribution can significantly reduce energy consumption, but they do not address dynamic stability. Particularly notable is the study by Wang *et al.* [19], which investigates the operation of an electromechanical drive with torque redistribution, explores different system topologies, and models their performance. While these approaches demonstrate high effectiveness, they are not applicable to vehicles with mechanical differentials, which demand fundamentally different control strategies.

The second group of studies [20], [21], [22], [23] focuses on off-road vehicles but relies either on empirical black-box modeling methods [20] or on simplified quasi-stationary models [21], which fail to capture essential dynamic transitional processes. For example, Mutha *et al.* [21] demonstrated the effectiveness of torque vectoring on loose soils, yet their approach is based on a quasi-stationary model that neglects transient dynamics. Ivanov *et al.* [22] described an adaptive torque vectoring controller for marshy terrain but did not provide a formal mathematical framework, relying instead on neural networks without analyzing stability. In contrast, Zhou *et al.* [23] developed a comprehensive nonlinear model, but its practical application is limited by excessive computational demands. Particularly noteworthy is the work of Eberhart *et al.* [14], which applies Bogdanov–Takens bifurcation analysis to investigate stability. While this approach is theoretically well grounded, its practical implementation is constrained by the need for high-performance computing systems. Similar challenges arise with modern algorithms based on deep reinforcement learning [24], [25], which demonstrate strong performance but demand extensive training and significant computational resources. Overall, the reviewed studies reveal a clear gap in stability control for vehicles with mechanical differentials. None of the existing approaches provides a comprehensive solution that combines rigorous analytical stability criteria, the ability to account for non-stationary motion modes, practical applicability to conventional transmissions, and moderate computational requirements.

Overall, the reviewed studies reveal a clear gap in the field of stability control for vehicles equipped with mechanical differentials. None of the existing approaches provides a

comprehensive framework that simultaneously offers rigorous analytical stability criteria, the capability to account for nonstationary driving regimes, practical applicability to conventional drivetrains, and moderate computational demands.

To address this gap, the present study employs Lyapunov stability theory. This method was chosen because it directly overcomes the identified limitations:

- it provides analytical guarantees for nonlinear systems without relying on numerical integration;
- it remains effective in the presence of strong nonlinearities, enabling the modeling of nonstationary dynamics;
- it yields explicit, verifiable conditions for control parameters, making it suitable for real-world transmission systems;
- its computational requirements are moderate, eliminating the need for real-time optimization or large training datasets.

Current study proposes a new approach aimed to fill this gap. It is based on the theory of motion stability developed by Lyapunov and represents a mathematical model that connects torque redistribution between axles and wheels with stability through Lyapunov functions, including non-stationary modes (acceleration/braking while making a turn). The model formulates simple verifiable conditions for control parameters that are feasible in real transmissions and proposes adaptive strategies for different scenarios (understeer/oversteer), combining differential locking and wheel braking.

## 2. Method

### 2.1. Research Subject

The study's subject is a 4x4 vehicle with mechanical inter-axle and inter-wheel differentials, and with the possibility of dynamic torque redistribution between the axles and the wheels of the front/rear axles. The key parameters are shown in Table 1.

Additional measures to enhance the stabilization of dual-axle vehicle motion, beyond torque redistribution between the driving axles, include:

- Redistributing torque between the wheels of an axle using controllable inter-wheel differentials;

**Table 1.** The key parameters

| Parameter                                            | Value                                                                                        |
|------------------------------------------------------|----------------------------------------------------------------------------------------------|
| $L$                                                  | : Wheelbase                                                                                  |
| $J_z$                                                | : Moment of inertia along axis $z$                                                           |
| $K_y$                                                | : Total (for the AWTVC axis) coefficient of wheel slip resistance                            |
| $\delta_A = \delta_2 - \delta_1$                     | : Difference between the slip angles of the front and rear axle wheels                       |
| $X_\Sigma = X_1 + X_2$                               | : Total traction force                                                                       |
| $X_1 = X_{1L} + X_{1R}$                              | : Total traction force on the front axle                                                     |
| $X_2 = X_{2L} + X_{2R}$                              | : Total traction force on the rear axle                                                      |
| $h_t$                                                | : The fraction of the total traction force transmitted by the transmission to the front axle |
| $\lambda_0 = \frac{M_f}{M_b}$                        | : Coefficient of asymmetry of the inter-axle differential                                    |
| $M_f, M_r$                                           | : Torques transmitted from the engine to the front and rear axles of the AWTVC, respectively |
| $\theta_{1av} = \frac{\theta_{1L} + \theta_{1R}}{2}$ | : Average turning angle of steerable wheels                                                  |
| $\delta_{iav} = \frac{\delta_{iL} + \delta_{iR}}{2}$ | : Average slip angle of wheels on the $i$ -th axle                                           |

- Redistributing torque between the wheels of an axle through individual wheel braking;
- Introducing additional resistance to turning by locking the inter-wheel differential in cases of rear-axle oversteer.

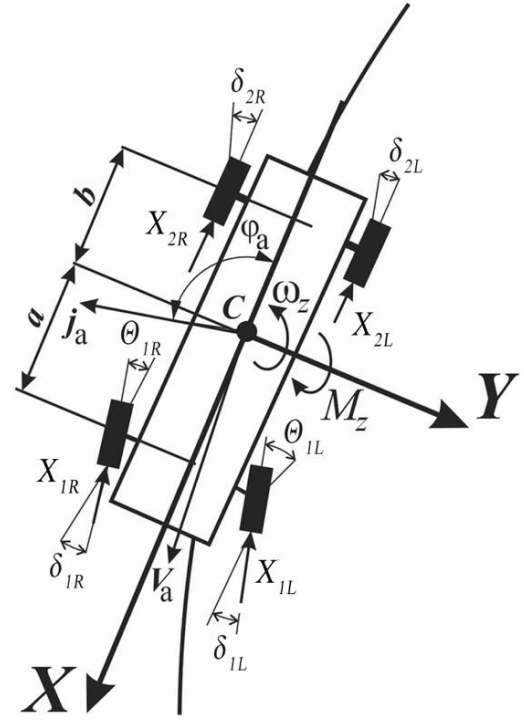
These possibilities are considered and discussed in detail in the following sections.

## 2.2. Mathematical Model

Lyapunov functions were selected to analyze the stability of a 4×4 vehicle because this method offers three key advantages. First, it allows for obtaining analytical stability guarantees for nonlinear systems with variable parameters without requiring numerical integration of the motion equations. Second, unlike frequency-domain methods, the Lyapunov approach remains applicable under strong nonlinearities in tire and transmission models. Third, it provides a constructive procedure for control synthesis: by defining a candidate function  $V(x)$  and its derivative  $V'(x)$ , it is possible to derive conditions for controller parameters directly, which is particularly important for systems with mechanical differentials.

### 2.2.1. Stability criteria for the redistribution of torque between axles

Figure 1 provides a diagram of the forces acting on a vehicle. For a 4×4 vehicle, the following differential equations were obtained for the front steerable wheels with torque redistribution between axles, considering the variable steering speed of the front axle steerable wheels are following Eq. (1) to Eq. (3).

**Figure 1.** Diagram of forces acting on a vehicle

where,  $C$  — vehicle center of mass;  $X, Y$  — axes of the coordinate system connected with the vehicle center of mass;  $a, b$  — distance from the vehicle center of gravity to the first and last axle, respectively;  $j_a$  — acceleration of the vehicle center of mass;  $X_{iL}, X_{iR}$  — longitudinal forces acting on, respectively, the left and right wheels of the  $i$ -th axle from the road;  $\theta_{1L}, \theta_{1R}$  — turning angles of the left and right steerable wheels, respectively;  $\delta_{iL}, \delta_{iR}$  — slip angles of the left and right wheels on the  $i$ -th axle;  $\phi_a$  — angle between the acceleration vector  $j_a$  and the longitudinal axis of a vehicle.

$$\dot{\delta}_1 = \frac{V_a}{L}(\theta_{1av} + \delta_2 - \delta_1) - \frac{K_y}{V_a}\left(\frac{g}{G_a} + \frac{L^2}{4J_z}\right)\delta_1 - \frac{K_y}{V_a}\left(\frac{g}{G_a} - \frac{L^2}{4J_z}\right)\delta_2 + \frac{j_a \cos \varphi_a}{V_a}(\theta_{1av} - \delta_1) - \frac{\theta_{1av}}{V_a}\left(\frac{g}{G_a} + \frac{L^2}{4J_z}\right)h_t X_\Sigma + \frac{d\theta_{1av}}{dt} \quad (1)$$

$$\dot{\delta}_2 = \frac{V_a}{L}(\theta_{1av} + \delta_2 - \delta_1) - \frac{K_y}{V_a}\left(\frac{g}{G_a} - \frac{L^2}{4J_z}\right)\delta_1 - \frac{K_y}{V_a}\left(\frac{g}{G_a} + \frac{L^2}{4J_z}\right)\delta_2 - \frac{j_a \cos \varphi_a}{V_a}\delta_2 - \frac{\theta_{1av}}{V_a}\left(\frac{g}{G_a} - \frac{L^2}{4J_z}\right)h_t X_\Sigma \quad (2)$$

$$\dot{\delta}_A = -\left(\frac{K_y}{V_a}\frac{L^2}{2J_z} + \frac{j_a \cos \varphi_a}{V_a}\right)\delta_A + \theta_{1av}\left(\frac{L^2}{2V_a J_z}h_t X_\Sigma - \frac{j_a \cos \varphi_a}{V_a}\right) - \frac{d\theta_{1av}}{dt}. \quad (3)$$

A stability analysis of the solution to Eq. (3) is carried out using the second Lyapunov method. According to it, a scalar function defined in vicinity of the equilibrium position has the following properties:

Function  $V(x)$  is above zero for all  $x \neq 0$  and  $V(0) = 0$ ;

The time derivative of function  $V(x)$  has to be non-positive for the same vicinity ( $V'(x) \leq 0$ ).

A candidate Lyapunov function is introduced in the form of:

$$V = \delta_A^2 + \theta_{1av}^2 > 0 \quad (4)$$

Differentiating the function  $V(\delta_A, \theta_{1av})$  with respect to time yields:

$$\begin{aligned} \frac{dV}{dt} &= \frac{\partial V}{\partial \delta_A} \frac{\partial \delta_A}{\partial t} + \frac{\partial V}{\partial \theta_{1av}} \frac{\partial \theta_{1av}}{\partial t} \\ &= \delta_A \dot{\delta}_A + \theta_{1av} \dot{\theta}_{1av}. \end{aligned} \quad (5)$$

The solution to Eq. (3) will be stable if the derivative of Lyapunov function (5) is non-positive.

Considering a stationary turn ( $\frac{d\theta_{1av}}{dt} = 0$ ), the derivative of Lyapunov function gives:

$$\frac{dV}{dt} = 2\delta_A \left\{ -\left(\frac{K_y L^2}{2J_z} + j_a \cos \varphi_a\right)\delta_A + \left(\frac{L^2}{2J_z}h_t X_\Sigma - j_a \cos \varphi_a\right)\theta_{1av} \right\}. \quad (6)$$

With neutral steering ( $\delta_A = 0$ ), expression (6) is non-positive with any  $h_t$ .

In the case of understeer (for  $\delta_A > 0$ ), from (6) follows the condition imposed on the value of  $h_t$ :

$$\frac{dV}{dt} = 2\delta_A \left\{ -\left(\frac{K_y L^2}{2J_z} + j_a \cos \varphi_a\right)\delta_A + \left(\frac{L^2}{2J_z}h_t X_\Sigma - j_a \cos \varphi_a\right)\theta_{1av} - \frac{d\theta_{1av}}{dt}V_a \right\} + 2\theta_{1av}V_a \frac{d\theta_{1av}}{dt}. \quad (9)$$

With neutral steering ( $\delta_A = 0$ ), expression (9) is non-positive with any  $h_t$ .

For understeer,

$$h_t \leq \frac{1}{X_\Sigma} \left\{ K_y \frac{\delta_A}{\theta_{1av}} + \frac{2J_z}{L^2} \left(1 + \frac{\delta_A}{\theta_{1av}}\right) j_a \cos \varphi_a + \frac{2J_z V_a}{L^2} \frac{\dot{\theta}_{1av}}{\theta_{1av}} \left(1 - \frac{\theta_{1av}}{\delta_A}\right) \right\} \quad (10)$$

For oversteer,

$$h_t \geq \frac{1}{X_\Sigma} \left\{ K_y \frac{\delta_A}{\theta_{1av}} + \frac{2J_z}{L^2} \left(1 + \frac{\delta_A}{\theta_{1av}}\right) j_a \cos \varphi_a + \frac{2J_z V_a}{L^2} \frac{\dot{\theta}_{1av}}{\theta_{1av}} \left(1 - \frac{\theta_{1av}}{\delta_A}\right) \right\}. \quad (11)$$

$$h_t \leq \frac{2}{X_\Sigma} \left\{ K_y \frac{\delta_A}{\theta_{1av}} + \frac{J_z}{L^2} \left(1 + \frac{\delta_A}{\theta_{1av}}\right) j_a \cos \varphi_a \right\}. \quad (7)$$

The limit on  $h_t$  for oversteer ( $\delta_A < 0$ )

$$h_t \geq \frac{2}{X_\Sigma} \left\{ K_y \frac{\delta_A}{\theta_{1av}} + \frac{J_z}{L^2} \left(1 + \frac{\delta_A}{\theta_{1av}}\right) j_a \cos \varphi_a \right\}. \quad (8)$$

Both conditions hold true at  $0 \leq h_t \leq 1$ . This means that changes in  $h_t$  within the physically possible limits do not lead to the vehicle losing stability in motion.

Considering a non-stationary turn ( $\frac{d\theta_{1av}}{dt} \neq 0$ ), the expression for  $\frac{dV}{dt}$  is:

An analysis of the order of magnitudes of the variables in the right-hand side of expressions (10) and (11) shows that it is always either, as in (10), or  $\ll 1$ , as in (11). From this it follows that both conditions are true for  $0 \leq h_t \leq 1$ . This means that changes in  $h_t$  within the physically possible limits do not cause a vehicle to lose stability in motion.

### 2.2.2. Stability criteria for torque redistribution between the axles and rear axle wheels

For 4x4 vehicle with torque redistribution between the axles and rear axle wheels, the equations of state (3) were considered again, and a candidate for the Lyapunov function in the form of (4) was introduced:

Under neutral steering ( $\delta_A = 0$ ), condition (12) is met with any  $\lambda_r$  and  $h_t$ .

$$\delta_A \left\{ - \left( \frac{K_y L^2}{2J_z V_a} + \frac{j_a \cos \varphi_a}{V_a} + \frac{1}{J_z \omega_z} [1 - h_t] \frac{B}{2} X_\Sigma \frac{\lambda_r - 1}{\lambda_r + 1} \right) \delta_A - \left( \frac{1}{J_z \omega_z} [1 - h_t] \frac{B}{2} X_\Sigma \frac{\lambda_r - 1}{\lambda_r + 1} - \frac{L^2}{2J_z} h_t X_\Sigma + \frac{j_a \cos \varphi_a}{V_a} \right) \theta_{1av} \right\} \leq 0. \quad (12)$$

For understeer, division both parts of (12) by  $\delta_A > 0$ , and after transforming, gives:

$$\frac{\lambda_r - 1}{\lambda_r + 1} \leq Q, \quad (13)$$

$$Q = \frac{J_z \omega_z}{[1 - h_t] \frac{B}{2} X_\Sigma} \times \left\{ \frac{j_a \cos \varphi_a}{V_a} + \frac{L^2}{2J_z V_a (\delta_A + \theta_{1av})} (K_y \delta_A - h_t X_\Sigma \theta_{1av}) \right\} > 0 \quad (14)$$

As a result, the shares of rear axle torque attributable to the lagging ( $h_{LAG\_R}$ ) and running ahead ( $h_{AH\_R}$ ) are to be determined as follows:

$$h_{LAG\_R} = \frac{\lambda_r}{\lambda_r + 1}; \quad h_{AH\_R} = \frac{1}{\lambda_r + 1} \quad (15)$$

$$h_{AH\_R} - h_{LAG\_R} \geq Q \quad (16)$$

In the case of oversteer (rear axle oversteer)  $\delta_A > 0$

$$\frac{\lambda_r - 1}{\lambda_r + 1} \geq Q < 0, \quad (17)$$

$$h_{AH\_R} - h_{LAG\_R} \leq Q \quad (18)$$

As a result, a traction drive control strategy for a 4x4 vehicle with torque redistribution between the axles and rear axle wheels can be formulated. In the case of rear axle oversteer, greater torque should be transmitted to the rear axle wheel

closest to the center of the turn. In this process,  $h_t$  needs to be decreased while satisfying the condition of  $h_t < 1$ , i.e., maintaining torque on the rear wheels. On the other hand, to increase stability in understeer, more torque has to be transferred to the rear axle wheel on the outer side relative to the center of rotation, or, which is the same, breaking the rear axle wheel closest to the center of rotation. In this case,  $h_t$  needs to be decreased while observing the condition of  $h_t > 0$ , i.e., keeping the front axle wheels in traction mode.

### 2.2.3. Stability criteria for torque redistribution between the axles and front axle wheels

For a vehicle with 4x4 wheel formula and torque redistribution between the axles and front axle wheels, the condition imposed on the total time derivative of Lyapunov functions can be written as Eq. (19).

$$\delta_A \left\{ - \left( \frac{K_y L^2}{2J_z V_a} + \frac{j_a \cos \varphi_a}{V_a} + \frac{1}{J_z \omega_z} h_t \frac{B}{2} X_\Sigma \frac{\lambda_f - 1}{\lambda_f + 1} \right) \delta_A - \left( \frac{1}{J_z \omega_z} h_t \frac{B}{2} X_\Sigma \frac{\lambda_f - 1}{\lambda_f + 1} - \frac{L^2}{2J_z} h_t X_\Sigma + \frac{j_a \cos \varphi_a}{V_a} \right) \theta_{1av} \right\} \leq 0. \quad (19)$$

With neutral steering ( $\delta_A = 0$ ), condition (19) is satisfied with any  $\lambda_f$  and  $h_t$ .

In the case of understeer ( $\delta_A > 0$ ), after transforming (19):

$$-\frac{\lambda_f - 1}{\lambda_f + 1} \leq R, \quad (20)$$

$$R = \frac{J_z \omega_z}{h_t \frac{B}{2} X_\Sigma} \left\{ \frac{j_a \cos \varphi_a}{V_a} + \frac{L^2}{2J_z V_a (\delta_A + \theta_{1av})} (K_y \delta_A - h_t X_\Sigma \theta_{1av}) \right\} > 0 \quad (21)$$

As a result, the shares of front axle torque attributable to the lagging ( $h_{LAG\_F}$ ) and running ahead ( $h_{AH\_F}$ ) are to be determined as follows:

$$h_{LAG\_F} = \frac{\lambda_f}{\lambda_f + 1}; \quad h_{AH\_F} = \frac{1}{\lambda_f + 1} \quad (22)$$

$$h_{AH\_F} - h_{LAG\_F} \leq R \quad (23)$$

In the case of oversteer (rear axle oversteer),  $\delta_A < 0$

$$-\frac{\lambda_f - 1}{\lambda_f + 1} \geq R < 0, \quad (24)$$

$$h_{AH\_F} - h_{LAG\_F} \geq R \quad (25)$$

It makes possible to develop a strategy for controlling the traction drive of a vehicle with a 4x4 wheel formula and torque redistribution between the axles and front axle wheels.

### 2.3. Control Strategies

#### 2.3.1. Inter-axle torque redistribution

Considering the equation of state (3), in understeer (the rear axle lagging behind) greater torque needs to be transmitted to the rear axle of a vehicle to lower  $h_t$ . This can be written as:

$$h_t = \min \left\{ \frac{\lambda_0}{\lambda_0 + 1}; \frac{1}{\lambda_0 + 1} \right\}, \quad (26)$$

$$\lambda_0 = \frac{M_f}{M_r}, \quad (27)$$

where  $\lambda_0$  — coefficient of asymmetry of the inter-axle differential;  $M_f, M_r$  — torques transmitted from the engine to the front and rear axles, respectively.

Conversely, in the case of rear axle oversteer (the front axle lagging behind), greater torque should be transmitted to the front axle to increase  $h_t$ . Then:

$$h_t = \max \left\{ \frac{\lambda_0}{\lambda_0 + 1}; \frac{1}{\lambda_0 + 1} \right\}. \quad (28)$$

When designing automatic control systems, efforts are made to ensure their stability with some guarantee so that changes in system parameters during operation do not lead to system instability. One of the effective methods for studying the directional and trajectory stability of a vehicle as a multidimensional system is the

frequency method, which allows for an a priori assessment of the influence of vehicle design parameters and the parameters of its motion on stability [5].

Consideration of the transfer characteristic  $W_{12}(s)$  of a vehicle with an all-wheel drive and steering front wheels gives:

$$W_{12}(s) = \frac{\delta_A(s)}{\theta_{1av}(s)}; s = j\omega, \quad (29)$$

where  $s$  — the Laplace operator;  $j = \sqrt{-1}$  — the imaginary unit;  $\omega$  — circular frequency.

For our case:

$$W_{12}(j\omega) = \frac{\left[ \frac{L^2}{2V_a J_z} h_t X_\Sigma - \frac{j_a \cos \varphi_a}{V_a} \right] \left[ \frac{K_y L^2}{V_a 2J_z} + \frac{j_a \cos \varphi_a}{V_a} - j\omega \right]}{\omega^2 + \left[ \frac{K_y L^2}{V_a 2J_z} + \frac{j_a \cos \varphi_a}{V_a} \right]^2} \quad (30)$$

Setting the modulus of the transfer characteristic equal to zero ( $|W_{12}(j\omega)| = 0$ ), which corresponds to the neutral steering, expressions for  $h_t$  was obtained.

For a rear-wheel drive vehicle with a switchable front axle ( $h_t = [0 \dots 0.5]$ ,  $0 \leq \lambda_0 \leq 1$ ):

$$h_t = \frac{\lambda_0}{\lambda_0 + 1}, h_t \sim 1 - \cos \varphi_a; \quad (31)$$

the front axle lagging behind (rear axle oversteer):

$$h_t = \frac{1}{\lambda_0 + 1}, h_t \sim \cos \varphi_a. \quad (32)$$

For a front-wheel drive vehicle with a switchable rear axle ( $h_t = [1 \dots 0.5]$ ,  $1 \leq \lambda_0 \leq \infty$ ) the equation (32) works, while equation (31) describes front axle lagging behind (rear axle oversteer).

#### 2.3.2. Control of inter-wheel differentials of the rear axle

In this scenario, the transfer characteristic  $W_{12}(s)$  takes the following form:

$$W_{12}(j\omega) = \frac{\left[ \frac{L^2}{2V_a J_z} h_t X_\Sigma - \frac{j_a \cos \varphi_a}{V_a} - \frac{1}{J_z \omega_z} (1 - h_t) \frac{B}{2} X_\Sigma \frac{\lambda_3 - 1}{\lambda_3 + 1} \right]}{\omega^2 + \left[ \frac{K_y L^2}{V_a 2J_z} + \frac{j_a \cos \varphi_a}{V_a} + \frac{1}{J_z \omega_z} (1 - h_t) \frac{B}{2} X_\Sigma \frac{\lambda_3 - 1}{\lambda_3 + 1} \right]^2} \times \frac{\left[ \frac{K_y L^2}{V_a 2J_z} + \frac{j_a \cos \varphi_a}{V_a} + \frac{1}{J_z \omega_z} (1 - h_t) \frac{B}{2} X_\Sigma \frac{\lambda_3 - 1}{\lambda_3 + 1} - j\omega \right]}{\omega^2 + \left[ \frac{K_y L^2}{V_a 2J_z} + \frac{j_a \cos \varphi_a}{V_a} + \frac{1}{J_z \omega_z} (1 - h_t) \frac{B}{2} X_\Sigma \frac{\lambda_3 - 1}{\lambda_3 + 1} \right]^2} \quad (33)$$

Next, the square modulus of expression (33) was found and set to zero ( $|W_{12}(j\omega)|^2 = 0$ ), which corresponds to the neutral steering of a vehicle:

$$\left[ \frac{L^2}{2V_a J_z} h_t X_\Sigma - \frac{j_a \cos \varphi_a}{V_a} - \frac{1}{J_z \omega_z} (1 - h_t) \frac{B}{2} X_\Sigma \frac{\lambda_3 - 1}{\lambda_3 + 1} \right]^2 \times \left[ \left( \frac{K_y L^2}{V_a 2J_z} + \frac{j_a \cos \varphi_a}{V_a} + \frac{1}{J_z \omega_z} (1 - h_t) \frac{B}{2} X_\Sigma \frac{\lambda_3 - 1}{\lambda_3 + 1} \right)^2 + \omega^2 \right] = 0 \quad (34)$$

The second factor in parentheses in expression (34) is always greater than zero. Then, to determine  $\lambda_r$ :

$$\frac{L^2}{2V_a J_z} h_t X_\Sigma - \frac{j_a \cos \varphi_a}{V_a} - \frac{1}{J_z \omega_z} (1 - h_t) \frac{B}{2} X_\Sigma \frac{\lambda_3 - 1}{\lambda_3 + 1} = 0, \quad (35)$$

from which, given that  $\omega_z = \frac{V_a}{L} \theta_{1av}$  and taking  $j_a = 0$  (uniform motion):

$$\lambda_r = \frac{\frac{h_t L \theta_{1av}}{(1-h_t)B} + 1}{1 - \frac{h_t L \theta_{1av}}{2(1-h_t)B}}. \quad (36)$$

As a result, expressions for the share of torque (from the total torque transmitted from the engine on the rear drive axle) for the wheel running ahead ( $h_{AH\_R}$ ) and the one lagging behind ( $h_{LAG\_R}$ ) take the form:

$$h_{AH\_R} = \min \left\{ \frac{\lambda_r}{\lambda_r + 1}; \frac{1}{\lambda_r + 1} \right\}; \quad (37)$$

$$h_{LAG\_R} = \max \left\{ \frac{\lambda_3}{\lambda_r + 1}; \frac{1}{\lambda_r + 1} \right\}. \quad (38)$$

### 2.3.3. Differential locking

In rear axle oversteer, locking the front axle differential in addition to redistributing torques creates a moment of resistance  $M_{res1}$ , which simultaneously acts as a counter-torque, contributing to the stabilization of motion:

$$M_{res1} = \frac{B}{2} (R_{x1} + f R_{z1}); \quad (39)$$

where  $R_{x1}$ ,  $R_{z1}$  — summed reactions on the front axle wheels in projections on the X and Z axes of the body-fixed coordinate system (see [Figure 1](#)), respectively,  $f$  — rolling resistance coefficient. The algorithm for locking the inter-wheel differential of the front axle in the event of rear axle oversteer is as follows: if rear axle oversteer is diagnosed, the inter-wheel differential of the front axle is locked. Otherwise, it is unlocked.

## 3. Result and Discussion

To verify and test the developed mathematical model, a first-order driver reaction model was used, considering the driver's reaction delay (time for perception and action) and trajectory correction based on the current slip  $\delta_A$ , and limitations on the maximum speed of steering wheel rotation:

$$\tau \frac{d\theta_{1av}}{dt} + \theta_{1av} = K_p \delta_A + K_d \frac{d\delta_A}{dt}, \quad (40)$$

where  $\tau = 0.5$  — reaction time constant,  $K_p = 1.2$  — proportional feedback coefficient (sensitivity to slip),  $K_d = 0.3$  — differential feedback coefficient (damping).

Model validation was carried out through numerical simulations in MATLAB/Simulink. The vehicle parameters correspond to an all-wheel-drive two-axle platform: wheelbase  $L = 3.2$  m, yaw moment of inertia  $J_z = 3000$  kg·m<sup>2</sup>, and overall tire cornering stiffness  $K_y = 100.000$  N/rad. The simulations examined a step change in steering input followed by a steady-state cornering phase.

The initial velocity was set to  $V_a = 15$  m/s, while the longitudinal acceleration  $j_a$  was varied from  $-3$  m/s<sup>2</sup> (braking) to  $+2$  m/s<sup>2</sup> (acceleration) to reproduce non-stationary maneuvers. The parameter  $h_t$  was varied within its physical bounds  $[0, 1]$  to analyze the vehicle's response under understeering and oversteering conditions.

The model allows examining non-stationary turns. For the considered impact, the graphs of the difference in slip angles of the front and rear axle wheels  $\delta_A$  and the average turning angle of steerable wheels  $\theta_{1av}$  are shown in [Figure 2](#).

The plots show that at  $h_t = 0.8$ ,  $\delta_A$  tends towards a positive value ( $\sim 0.1$  rad), which corresponds to understeer, and the system is stable (there are no divergent oscillations). At  $h_t = 0.2$ ,  $\delta_A$  tends towards a negative value ( $\sim -0.04$  rad), which corresponds to oversteer. In this case, the system is still stable but close to the limit of stability. During oversteer, the transient process also demonstrates oscillatory behavior.

[Figure 3](#) demonstrates the dependencies of the difference between the slip angles of front and rear axle wheels  $\delta_A$  and the average turning angle of steerable wheels  $\theta_{1av}$  at different  $h_t$ .

[Figure 4](#) shows the phase portrait at a fixed value of  $h_t$ . The direction of the vectors indicates the stability of the function.

The phase portrait shown in [Figure 4](#) confirms the asymptotic stability of the equilibrium point, as all trajectories converge to the origin. The developed Lyapunov-based control strategy directly addresses the shortcomings identified in the literature review. Unlike quasistationary models or “black-box” methods, our approach provides an analytical framework for evaluating stability in transient regimes through the derivative ( $dV/dt$ ). Moreover, in contrast to computationally intensive NMPC (Nonlinear Model Predictive Control) methods, it yields simple inequalities (7), (8), (10), (11) for the control parameters, which are suitable for real-time implementation on standard automotive electronic

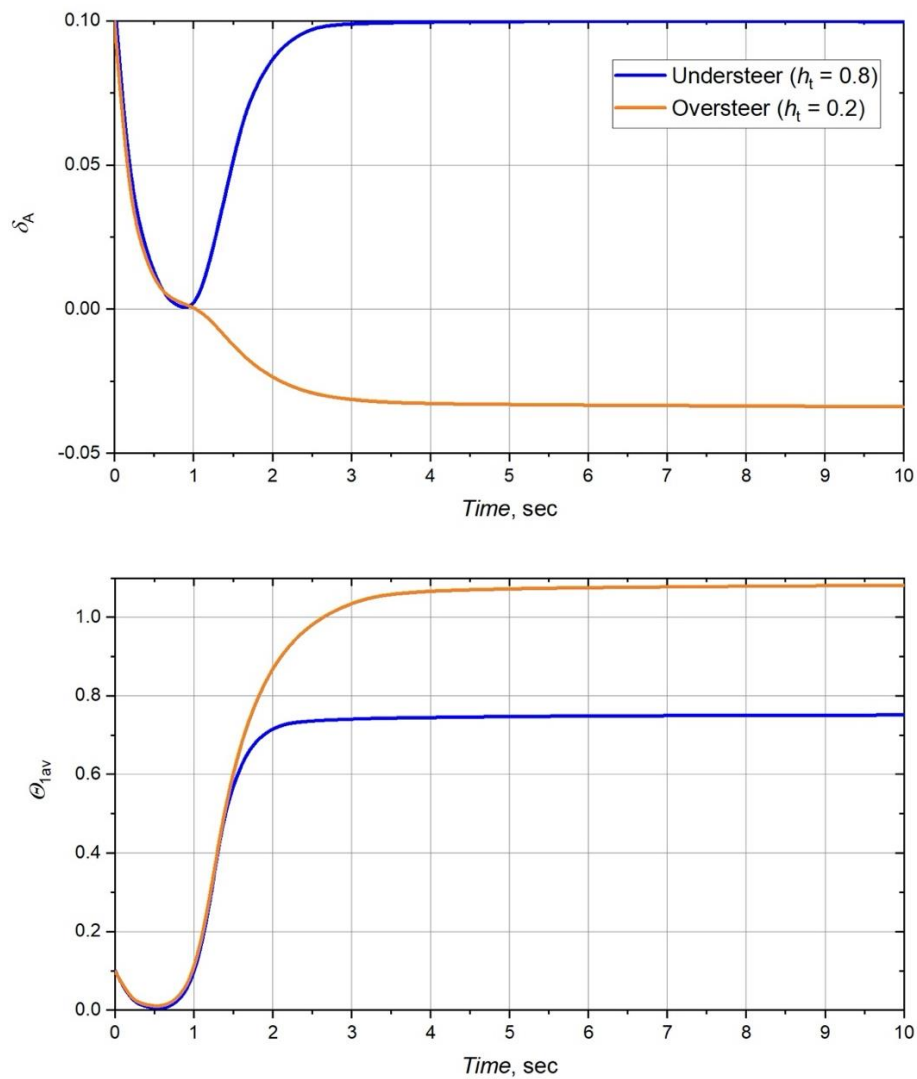


**Figure 2.** Dependence of the difference between the slip angles of front and rear axle wheels  $\delta_A$  and the average turning angle of steerable wheels  $\theta_{1av}$  on time

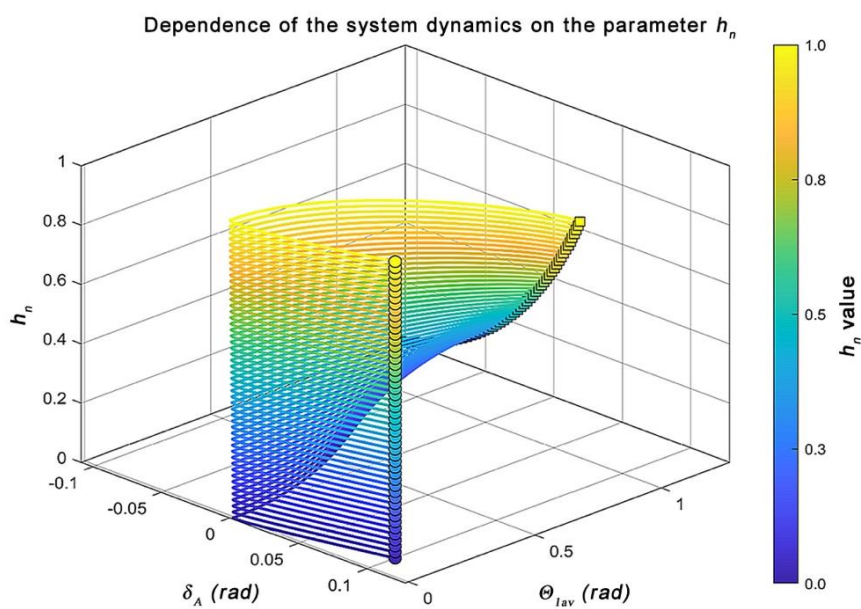


**Figure 3.** Dependencies of  $\delta_A$  and  $\theta_{1av}$  with different  $h_t$

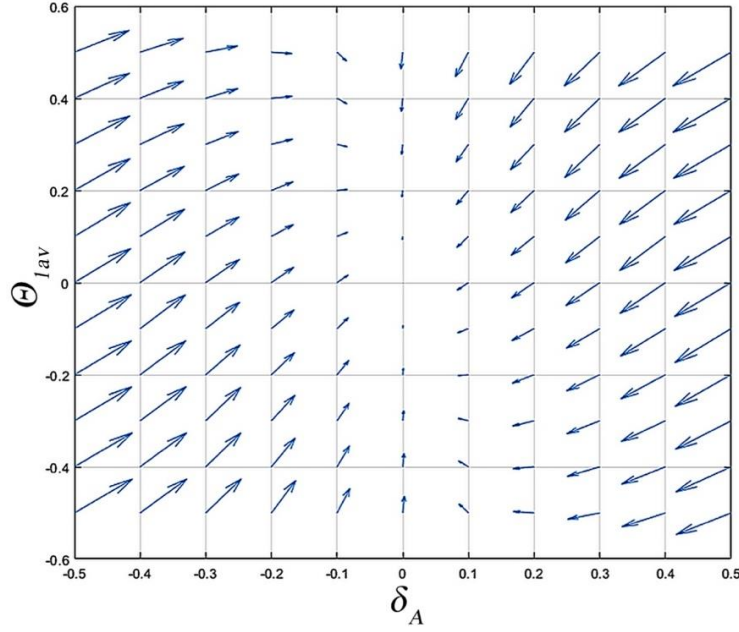


Figure 4. Phase portrait at  $h_t = 0.5$

control units. Furthermore, while research on electric-vehicle control or deep reinforcement learning is not directly applicable to mechanical drivetrains, our model introduces explicit criteria for torque redistribution via mechanical differentials and brake actuators, effectively filling the identified gap in the existing body of research.

It is worth noting a number of limitations of the described model, which include:

- Simplified tire dynamics — the model uses a linear relationship between slip and lateral force;
- Strict assumptions about driver influences — the chosen driver model is linear (a PD-controller), which does not account for adaptation to changing conditions (for example, transitioning from asphalt to ice);
- The longitudinal dynamics are not considered;
- Idealized torque redistribution — parameter  $h_t$  changes instantaneously, without considering the inertia of the transmission;
- External disturbances are not factored in — crosswinds, road irregularities, or changes in the coefficient of friction are not modeled.

Further development and upgrade of the proposed model will allow to take these aspects into account by introducing a tire-soil interaction model, modifying the driver control logic, introducing an engine and transmission model, and incorporating stochastic or deterministic disturbances.

Obtained results are consistent with findings of studies in the field, namely with Guo *et al.* [15] and Diez *et al.* [16] concerning torque redistribution, which shows that torque vectoring to the outer wheels improves steering stability. However, while these works focus on electric vehicles, the proposed model demonstrates a similar effect for mechanical differentials through parameter  $h_t$ . Moreover, obtained results are in agreement with the conclusions of Zhou *et al.* [23] about the important role of torque control in off-road wheeled vehicles, but instead of empirical methods, we propose an analytical solution. Considering stability in non-stationary regimes, the obtained conditions for the non-positive time derivative of Lyapunov functions ( $V'(x) \leq 0$ ) supplement the results of previous studies [14], [26], where stability was examined through bifurcation analysis or pitch control. The proposed approach requires fewer computational resources and offers excellent performance, which is especially important for transportation systems.

Talking about novelty of the proposed approach, three points can be highlighted.

Firstly, mechanical transmission criteria. In contrast to Jin *et al.* [24] and Caponio *et al.* [27] using Model Predictive Control (MPC) and Deep Deterministic Policy Gradients (DDPG), the introduced conditions (5)-(6) give explicit restrictions for  $h_t$  and  $\lambda$  that require no real-time optimization. This is consistent with the conclusions of Nussupbek *et al.* [28] about the

importance of simple verifiable conditions, but expands them to dynamic modes.

Secondly, accounting for inertia loads. While the studies of Ivanov *et al.* [22] and Zhou *et al.* [23] neglect transitional processes, our model includes the  $\frac{d\theta_{1av}}{dt}$  element, which is crucial for curvilinear motion with acceleration or deceleration maneuvers.

Thirdly, comparison with AI-based methods. Although the work of Jin *et al.* [24] showcases the advantages of DDGP for multi-axle vehicles, our approach shows comparable efficiency without the need for training neural networks, which simplifies its implementation.

Unlike more complex systems [15], [27], the proposed control strategy involves only measuring the slip angles ( $\delta_1, \delta_2$ ) and controlling parameters  $h_t$  and  $\lambda$  in real time. Thus, computational power requirements are lowered or directed towards increasing autonomy, which aligns with the concept of Agriculture 5.0, as described by Ghobadpour *et al.* [29]. Similar to Jin *et al.* [24], our model does not account for changes in the friction coefficient, which may require the algorithms to be adapted to complex surfaces such as snow, ice, or sand. However, unlike a black-box model, our method allows for integrating additional conditions easily by modifying Lyapunov functions.

Thus, the proposed model allows for precise calculation of the redistribution of torque to the inner or outer wheels when a vehicle skids or understeers, respectively, to stabilize its movement without requiring much computing resources. The model could supplement existing stabilization systems (such as ESP) by ensuring stability in non-stationary modes through adaptive moment redistribution. As opposed to empirical methods [15], [16], [27], [30], the presented approach guarantees stability through Lyapunov functions, as confirmed by conditions (5)-(6).

## 4. Conclusion

Mathematical model for the stability of curvilinear motion of a vehicle with dynamic torque redistribution between axles and wheels was proposed. The model is based on the analysis of Lyapunov functions and allows for the formalization of stability conditions for various

modes of movement, including non-stationary scenarios (acceleration and braking in a turn). The study has demonstrated that through adaptive control of transmission elements, torque control strategies ensure stabilization.

The key contribution of the research is the integration of torque vectoring control methods with stability guarantees through Lyapunov functions, which has not previously been applied to vehicles with mechanical differentials. Unlike other existing solutions, the model considers the dynamic change in adherence conditions and inertia loads and offers simple, verifiable conditions for control parameters that can be used in real transmissions.

The developed model can be integrated into the active safety systems of 4x4 vehicles, including adaptive differential locking algorithms and wheel braking. A promising direction is to connect the model with MPC algorithms. In addition, the model can be expanded for three-axle and four-axle chassis by introducing additional equations. This will provide an opportunity to use a highly efficient and fast model for tasks involving the modeling and control of multi-axle chassis.

Future research will involve experimental validation at a proving ground using an all-wheel-drive vehicle instrumented with sensors capable of recording key motion parameters, including vehicle speed, steering-wheel angle, slip angle, and related quantities. Such experiments will enable comparison of the predicted stability boundaries with the actual vehicle response on surfaces with differing friction levels. The model is also suitable for adaptation to multi-axle chassis.

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## Author's Declaration

### Authors' contributions and responsibilities

The authors made substantial contributions to the conception and design of the study. The authors took responsibility for data analysis, interpretation and discussion of results. The authors read and approved the final manuscript.

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## Availability of data and materials

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The authors declare no competing interest.

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